The third order asymptotics admissibility of estimators in one parameter regular case

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The third order asymptotic admissibility of estimators in one parameter regular case

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$X_1, X_2, \cdots X_n$ はi.i.d.で共通の確率密度関数 $f(x, \theta_0)$, $\theta_0 \in \Theta \subset R$ を持つものとする。$f(x, \theta_0)$ は次の条件を満たす.

A1. Θ は R の開集合である.

A2. $f(x, \theta)$ は (x, θ) に関して可測である.

A3. 各 x に対して, $f(x, \theta)$ は θ に関して5回連続偏微分可能で, その微分係数は θ に関して連続である.

A4. $\{f(x, \theta), \theta \in \Theta\}$ に対応する測度は互いに絶対連続である.

A5. すべての $\theta \in \Theta$ に対して,

$$E_\theta |\log f(X, \theta)| < \infty$$

$$0 < I_{1,1}(\theta) = E_\theta\left(\frac{\partial}{\partial\theta}\log f(X, \theta)\right)^2 = -E_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f(X, \theta)\right).$$

A6. 各 $\theta \in \Theta$ に対して, コンパクトな θ_0 の近傍 Θ_0 および関数 $G(x)$ が存在して, すべての $\theta \in \Theta_0$ に対して

$$\left|\frac{\partial^i}{\partial\theta^i}\log f(X, \theta)\right| \leqq G(X), \quad i = 1, 2, 3, 4, 5.$$

$$\sup_{\theta \in \Theta_0} E_\theta (G(X))^4 < \infty.$$

A 7.　すべての $j\ (\leqq 5)$, $j = i_1 + i_2 + \cdots + i_5$, $i_j = 0, 1, 2, \cdots, 5$,

に対して,

$$E_\theta\left\{\frac{\partial^{i_1}}{\partial\theta^{i_1}}\log f(x,\theta)\cdot\frac{\partial^{i_2}}{\partial\theta^{i_2}}\log f(x,\theta)\cdot\cdots\cdot\frac{\partial^{i_5}}{\partial\theta^{i_5}}\log f(x,\theta)\right\}$$

は $\theta \in \Theta$ に関して, 4回連続微分可能で, その微分係数は θ に関して連続である.

A 8.　θ の最尤推定量 $\hat{\theta}$ が一意に定まる.

すべての $\theta_0 \in \Theta$ に対して θ_0 の近傍 Θ_0 が存在して, すべての $\theta \in \Theta_0$ で一様に $o(n^{-1})$ まで Edgeworth 展開可能である θ の推定量 T の全体を $\mathfrak{F}$ とするとき以下の定理が成り立つ.

定理　推定量 $T \in \mathfrak{F}$ に対して, 関数 $c_0(\theta)$, $c_1(\theta)$, $t(\theta)$, $c_2(\theta, t(\theta))$ が存在して, 統計量

$$\begin{aligned}\bar{\theta} = \hat{\theta} &+ \frac{1}{n^{1/2}}c_0(\hat{\theta}) \\ &+ \frac{1}{n}\left\{\left(c_0(\hat{\theta}) - \frac{1}{2}t(\hat{\theta})\right)I_{1,1}^{-1}(\hat{\theta})Z_2(\hat{\theta}) + c_1(\hat{\theta})\right\} \\ &+ \frac{1}{n^{3/2}}\left\{\left(c_0(\hat{\theta}) - \frac{1}{2}t(\hat{\theta})\right)I_{1,1}^{-2}(\hat{\theta})\left(K_{2,2}(\hat{\theta}) - I_{1,1}^{2}(\hat{\theta})\right)\right. \\ &\qquad\qquad \left. + c_2(\hat{\theta}, t(\hat{\theta}))\right\}\end{aligned}$$

は次式を満たす:

すべての $x_1, x_2 \geqq 0$ に対して,

$$P_{\theta_0}(-x_1 \leqq n^{1/2}(T-\theta) \leqq x_2)$$
$$\leqq P_{\theta_0}(-x_1 \leqq n^{1/2}(\bar{\theta}-\theta) \leqq x_2) + o(n^{-1})$$

ただし

$$Z_2(\theta) = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial\theta^2} \log f(x,\theta) + I_{1.1}(\theta) \right)$$

$$K_{2.2}(\theta) = E_\theta \left(\frac{\partial^2}{\partial\theta^2} \log f(x,\theta) \cdot \frac{\partial^2}{\partial\theta^2} \log f(x,\theta) \right).$$

定理の証明

$$W_n(\theta, t) = \sum_{i=1}^{n} \log f(x_i, \theta + \frac{t}{n^{1/2}}) - \sum \log f(x_i, \theta) - b(\theta, t, \frac{1}{n^{1/2}})$$

とし，$W_n(\theta, t) = 0$ の解を $\theta^*(t)$ とすれば $C_0(\theta, t)$, $C_1(\theta, t)$, $C_2(\theta, t)$ が存在して，確率 $1 - o_p(n^{-1})$ で，

$$\begin{aligned}
\theta^*(t) = {} & \theta + \frac{1}{n^{1/2}}(I_{1.1}^{-1}(\theta)Z_1(\theta) + C_0(\theta, t) - t) \\
& + \frac{1}{n} \{ I_{1.1}^{-2}(\theta)Z_1(\theta)Z_2(\theta) + \frac{1}{2} I_{1.1}^{-3}(\theta)J_3(\theta)Z_1^2(\theta) \\
& + (C_0(\theta, t) - \frac{t}{2}) I_{1.1}^{-1}(\theta)(Z_2(\theta) - I_{1.1}^{-1}(\theta)J_{2.1}(\theta)Z_1(\theta)) \\
& + C_0'(\theta, t) I_{1.1}^{-1}(\theta)Z_1(\theta) + C_1(\theta, t) \} \\
& + \frac{1}{n^{3/2}} [I_{1.1}^{-3}(\theta)Z_1(\theta)Z_2^2(\theta) + \frac{3}{2} I_{1.1}^{-4}(\theta)J_3(\theta)Z_1^2(\theta)Z_2(\theta) \\
& + \frac{1}{2} I_{1.1}^{-3}(\theta)Z_1^2(\theta)Z_3(\theta) + \{ \frac{1}{2} I_{1.1}^{-5}(\theta)J_3^2(\theta) + \frac{1}{6} I_{1.1}^{-4}(\theta)K_4(\theta) \} Z_1^3(\theta) \\
& + \{ -(C_0(\theta, t) - \frac{t}{2}) I_{1.1}^{-3}(\theta)(5J_{2.1}(\theta) + J_{1.1.1}(\theta)) \\
& + 2C_0'(\theta, t) I_{1.1}^{-2}(\theta) \} Z_1(\theta)Z_2(\theta) + (C_0(\theta, t) - \frac{t}{2}) I_{1.1}^{-2}(\theta)Z_1(\theta)Z_3(\theta) \\
& + \{ \frac{1}{2}(C_0(\theta, t) - \frac{t}{2}) I_{1.1}^{-4}(\theta)J_{2.1}(\theta)(-J_3(\theta) + 6J_{2.1}(\theta) + 2J_{1.1.1}(\theta))
\end{aligned}$$

$$-\frac{1}{2}(C_0(\theta,t)-\frac{t}{2})I_{1.1}^{-3}(\theta)(2K_{3.1}(\theta)+K_{2.2}(\theta)+K_{2.1.1}(\theta))$$
$$+\frac{1}{2}C_0'(\theta,t)I_{1.1}^{-3}(\theta)(J_3(\theta)-2J_{2.1}(\theta))$$
$$+\frac{1}{2}C_0''(\theta,t)I_{1.1}^{-2}(\theta)\}Z_1^2(\theta)+(C_0(\theta,t)-\frac{t}{2})I_{1.1}^{-2}(\theta)Z_2^2(\theta)$$
$$+\{(C_0(\theta,t)-\frac{t}{2})C_0'(\theta,t)-(C_0(\theta,t)-\frac{t}{2})^2I_{1.1}^{-1}(\theta)J_{2.1}(\theta)+C_1(\theta,t)\}$$
$$\times I_{1.1}^{-1}(\theta)(Z_2(\theta)-I_{1.1}^{-1}(\theta)J_{2.1}(\theta)Z_1(\theta))+C_1'(\theta,t)I_{1.1}^{-1}(\theta)Z_1(\theta)$$
$$+\frac{1}{6}I_{1.1}^{-1}(\theta)\{3t(C_0(\theta,t)-t)+3(C_0(\theta,t)-t)^2+t^2\}$$
$$\times(Z_3(\theta)-I_{1.1}^{-1}(\theta)K_{3.1}(\theta)Z_1(\theta))+C_2(\theta,t)].$$

m.l.e. の言葉で言い替えれば,

$$\theta^*(t)=\hat\theta+\frac{1}{n^{1/2}}(C_0(\hat\theta,t)-t)$$
$$+\frac{1}{n}\{(C_0(\hat\theta,t)-\frac{t}{2})I_{1.1}^{-1}(\hat\theta)Z_2(\hat\theta)+C_1(\hat\theta,t)\}$$
$$+\frac{1}{n^{3/2}}[(C_0(\hat\theta,t)-\frac{t}{2})I_{1.1}^{-2}(\hat\theta)Z_2^2(\hat\theta)$$
$$+I_{1.1}^{-1}(\hat\theta)\{-(C_0(\hat\theta,t)-\frac{t}{2})^2I_{1.1}^{-1}(\hat\theta)J_{2.1}(\hat\theta)$$
$$+(C_0(\hat\theta\ t)-\frac{t}{2})C_0'(\hat\theta,t)$$
$$+C_1(\hat\theta,t)\}Z_2(\hat\theta)$$
$$+\frac{1}{6}\{3t(C_0(\hat\theta,t)-t)+3(C_0(\hat\theta,t)-t)^2+t^2\}$$
$$\times I_{1.1}^{-1}(\hat\theta)Z_3(\hat\theta)+C_2(\hat\theta,t)]+o_p(n^{-1}).$$

ところで

$$\theta^{**}(t)=\hat\theta+\frac{1}{n^{1/2}}(C_0(\hat\theta,t)-t)$$
$$+\frac{1}{n}\{(C_0(\hat\theta,t)-\frac{t}{2})I_{1.1}^{-1}(\hat\theta)Z_2(\hat\theta)+C_1(\hat\theta,t)\}$$
$$+\frac{1}{n^{3/2}}\{(C_0(\hat\theta,t)-\frac{t}{2})I_{1.1}^{-2}(\hat\theta)(K_{2.2}(\hat\theta)-I_{1.1}^{2}(\hat\theta))$$
$$+C_2(\hat\theta,t)\}$$

なる統計量を導入すれば,

$$\theta^*(t) = \theta^{**}(t) + o_p(n^{-1}) \quad (\textit{in law})$$

であることが分かる。

今推定量Tが，すべての $\theta \in \Theta$ に対して，

$$P_\theta(T \leqq \theta) = g(\theta, \frac{1}{n^{1/2}}) = 1 - P_\theta(T \geqq \theta)$$

を満たすとしよう．このとき，$\theta^{**}(t)$ が，任意の $\theta \in \Theta$ に対し，

$$P_{\theta + t/n^{1/2}}(\theta^{**}(t) \leqq \theta) \leqq g(\theta + \frac{1}{n^{1/2}}, \frac{1}{n^{1/2}}) + o(n^{-1})$$

を満たすならば，不等式

$$P_\theta(T \leqq \theta + \frac{t}{n^{1/2}}) \leqq P_\theta(\theta^{**}(t) \leqq \theta) + o(n^{-1}) \quad (t>0)$$

$$P_\theta(T \geqq \theta + \frac{t}{n^{1/2}}) + o(n^{-1}) \geqq P_\theta(\theta^{**}(t) \geqq \theta) \quad (t<0)$$

が成り立つ．

従って $\hat{\theta}(t) = \theta^{**}(t) + \frac{t}{n^{1/2}}$ と置けば，任意の $\theta \in \Theta$ に対して，

$$P_\theta(T \leqq \theta + \frac{t}{n^{1/2}}) \leqq P_\theta(\hat{\theta}(t) \leqq \theta + \frac{t}{n^{1/2}}) + o(n^{-1}) \quad (t>0)$$

$$P_\theta(T \geqq \theta + \frac{t}{n^{1/2}}) + o(n^{-1}) \geqq P_\theta(\hat{\theta}(t) \geqq \theta + \frac{t}{n^{1/2}}) \quad (t<0)$$

となる．

$T \in \mathcal{F}$ に対して，$\theta \in \Theta_0$ における $n^{\frac{1}{2}}(T-\theta)$ の i-th cumulant

$K_i(\theta)$ を

$$K_1(\theta, \frac{1}{n^{1/2}}) = \frac{1}{n^{3/2}}K_{11}(\theta) + \frac{1}{n}K_{12}(\theta) + o(n^{-1}),$$

$$K_2(\theta, \frac{1}{n^{1/2}}) = I_{1\cdot 1}^{-1}(\theta) + \frac{1}{n^{1/2}}K_{21}(\theta) + \frac{1}{n}K_{22}(\theta) + o(n^{-1}),$$

$$K_3(\theta, \frac{1}{n^{1/2}}) = \frac{1}{n^{1/2}}K_{31}(\theta) + \frac{1}{n}K_{32}(\theta) + o(n^{-1}),$$

$$K_4(\theta, \frac{1}{n^{1/2}}) = \frac{1}{n}K_{42}(\theta) + o(n^{-1}),$$

$$K_i(\theta) = o(n^{-1}) \quad (i \geqq 5)$$

で表わそう. $T \in \mathfrak{F}$ であるから, T に対応して定まる $\hat{\theta}(t)$ において $C_0(\theta, t)$, $C_1(\theta, t)$ は t と無関係であるから, 以後 $C_0(\theta, t)$, $C_1(\theta, t)$ に対して, $C_0(\theta)$, $C_1(\theta)$ を用いる.

ところで推定量 $T \in \mathfrak{F}$ が *second order admissible* であるならば

$$K_{10}(\theta) = C_0(\theta), \qquad K_{20}(\theta) = I_{1\cdot 1}^{-1}(\theta),$$

$$K_{11}(\theta) = C_1(\theta) - \frac{1}{2} I_{1\cdot 1}^{-2}(\theta)(J_{2\cdot 1}(\theta) + J_{1\cdot 1\cdot 1}(\theta)),$$

$$K_{21}(\theta) = 2 I_{1\cdot 1}^{-1}(\theta) K_{10}'(\theta),$$

$$K_{31}(\theta) = -I_{1\cdot 1}^{-3}(\theta)(3J_{2\cdot 1}(\theta) + 2J_{1\cdot 1\cdot 1}(\theta)),$$

ただし $K_{10}'(\theta) = \frac{d}{d\theta} K_{10}(\theta)$ かつ すべての $t \neq 0$ に対して,

$$\begin{aligned}
&I_{1\cdot 1}^{5/2}(\theta)\{-A_{31}(\theta) + (K_{42}^{*}(\theta) - K_{42}(\theta)) I_{1\cdot 1}(\theta)\} t^2 \\
&+ I_{1\cdot 1}^{5/2}(\theta)\ [4(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) \\
&\quad + \{4A_{31}(\theta) - 3(K_{42}^{*}(\theta) - K_{42}(\theta)) I_{1\cdot 1}(\theta)\} C_0(\theta)]\, t \\
&+ I_{1\cdot 1}^{3/2}(\theta)\ [12(A_{20}(C_0'(\theta), C_1'(\theta), \theta) - K_{22}(\theta)) - 3(K_{42}^{*}(\theta) - K_{42}(\theta)) I_{1\cdot 1}(\theta) \\
&\qquad - 8(A_{30}(C_0'(\theta), C_0''(\theta), \theta) - K_{32}(\theta)) I_{1\cdot 1}(\theta) C_0(\theta) \\
&\qquad + \{-4A_{31}(\theta) + 3(K_{42}^{*}(\theta) - K_{42}(\theta)) I_{1\cdot 1}(\theta)\} I_{1\cdot 1}(\theta) C_0^2(\theta)] \leqq 0
\end{aligned}$$

ただし

$$\begin{aligned}
A_{20}(C_0'(\theta), C_1'(\theta), \theta) = &-I_{1\cdot 1}^{-3}(\theta)(K_{3\cdot 1}(\theta) + 4K_{2\cdot 1\cdot 1}(\theta) + K_{1\cdot 1\cdot 1\cdot 1}(\theta) + I_{1\cdot 1}^{2}(\theta)) \\
&+ \frac{1}{2} I_{1\cdot 1}^{-4}(\theta)(7J_{2\cdot 1}^{2}(\theta) + 14J_{2\cdot 1}(\theta) J_{1\cdot 1\cdot 1}(\theta) + 5J_{1\cdot 1\cdot 1}^{2}(\theta)) \\
&+ (C_0'(\theta))^2 I_{1\cdot 1}^{-1}(\theta) + 2C_1'(\theta) I_{1\cdot 1}^{-1}(\theta),
\end{aligned}$$

$$\begin{aligned}
A_{30}(C_0'(\theta), C_0''(\theta), \theta) = &-3C_0'(\theta) I_{1\cdot 1}^{-3}(\theta)(3J_{2\cdot 1}(\theta) + 2J_{1\cdot 1\cdot 1}(\theta)) \\
&+ 3C_0''(\theta) I_{1\cdot 1}^{-2}(\theta),
\end{aligned}$$

$$A_{31}(\theta) = 3I_{1\cdot 1}^{-4}(\theta)\ \{I_{1\cdot 1}(\theta)(K_{2\cdot 2}(\theta) - I_{1\cdot 1}^{2}(\theta)) - J_{2\cdot 1}^{2}(\theta)\}.$$

$$\begin{aligned}
K_{42}^{*}(\theta) = &-I_{1\cdot 1}^{-4}(\theta)(4K_{3\cdot 1}(\theta) + 12K_{2\cdot 1\cdot 1}(\theta) + 3K_{1\cdot 1\cdot 1\cdot 1}(\theta) + 3I_{1\cdot 1}^{2}(\theta)) \\
&+ 12 I_{1\cdot 1}^{-5}(\theta)(2J_{2\cdot 1}(\theta) + J_{1\cdot 1\cdot 1}(\theta))(J_{2\cdot 1}(\theta) + J_{1\cdot 1\cdot 1}(\theta)))
\end{aligned}$$

上の不等式から

$K^*_{42}(\theta)-K_{42}(\theta)\leqq 0.$

$$\frac{1}{4}I_{1.1}(\theta)\{A_{31}(\theta)-(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}A^2(\theta)$$
$$+12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))-3(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)$$
$$-8(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))I_{1.1}(\theta)C_0(\theta)$$
$$+\{-4A_{31}(\theta)+3(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}I_{1.1}(\theta)C_0^2(\theta)\leqq 0,$$

ただし

$$A(\theta)=\frac{4(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))+(4A_{31}(\theta)-3(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta))C_0(\theta)}{-A_{31}(\theta)+(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)}$$

$t(\theta)=-\frac{1}{2}A(\theta)$ とおき, $\tilde{\theta}(t(\hat{\theta}))$とする。このとき,

$$\tilde{\theta}(t(\hat{\theta}))=\tilde{\theta}(t(\theta))+o_p(n^{-1})\quad(\textit{in law})$$

であって,

$$P_\theta(n^{1/2}(T-\theta)\leqq x)-P_\theta(n^{1/2}(\tilde{\theta}(t(\hat{\theta}))-\theta)\leqq x)$$
$$=\frac{1}{24n}\phi(I_{1.1}^{1/2}(\theta)(x-C_0(\theta)))I_{1.1}^{5/2}(\theta)x$$
$$\times[(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)(x+\frac{1}{2}A(\theta))^2$$
$$-\frac{1}{4}\{-A_{31}(\theta)+(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}A^2(\theta)$$
$$+12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))I_{1.1}^{-1}(\theta)-3(K^*_{42}(\theta)-K_{42}(\theta))$$
$$-8(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))C_0(\theta)$$
$$-\{4A_{31}(\theta)-3(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}C_0^2(\theta)]+o(n^{-1})$$

従って,

$$P_\theta(n^{1/2}(T-\theta)\leqq x)\leqq P_\theta(n^{1/2}\{\tilde{\theta}(t(\hat{\theta}))-\theta\}\leqq x)+o(n^{-1})\quad(x\geqq 0)$$

$$P_\theta(n^{1/2}(T-\theta)<x)+o(n^{-1})\geqq P_\theta(n^{1/2}\{\tilde{\theta}(t(\hat{\theta}))-\theta\}<x)\quad(x\leqq 0)$$

ゆえにすべての $x_1,x_2\geqq 0$ に対して,

$$P_\theta(-x_1\leqq n^{1/2}(T-\theta)\leqq x_2)\leqq P_\theta(-x_1\leqq n^{1/2}(\tilde{\theta}(t(\hat{\theta}))-\theta)\leqq x_2)+o(n^{-1})$$

が成り立つ.